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An exact solution of the one-dimensional Dirac oscillator in the presence of minimal lengths

Khireddine Nouicer

Department of Physics and Laboratory of Theoretical Physics (LPTh), Faculty of Sciences, University of Jijel, BP 98, Ouled Aissa, 18000 Jijel, Algeria

E-mail: khnouicer@mail.univ-jijel.dz

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Abstract
Using the momentum space representation, we determine the energy eigenvalues, eigenfunctions and the high-temperature thermodynamic properties of the Dirac oscillator in one dimension in the presence of a minimal length given by $(\Delta X)_{\min} = \hbar\sqrt{\beta}$, where β is the deformation parameter of the modified commutation relation $[X, P] = \mathrm{i}\hbar(1 + \beta P^2)$. The obtained results suggest that the effect of the minimal length could be detected in ultrarelativistic heavy-ion collisions.

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1. Introduction

In a series of papers, Kempf *et al* [1–4] introduced a deformed quantum mechanics based on modified commutation relations $[X_i, P_j] = \mathrm{i}\hbar[(1 + \beta P^2)\delta_{ij} + \beta' P_i P_j]$. These commutations relations lead to a generalized Heisenberg uncertainty (GUP) which define non-zero minimum uncertainties in position and/or momentum. A non-zero minimum position uncertainty or minimal length has first appeared in the context of perturbative string theory [5–7]. One major feature of this finding is that the physics below such a scale becomes inaccessible and then defining a natural cut-off which prevents from the usual UV divergences. The other consequence of such a GUP is the appearance of an intriguing UV/IR mixing. This mixing between UV and IR divergences, first noted in the ADS/CFT correspondence [8, 9], is also a feature of non-commutative quantum field theory [10, 11]. Physically, the UV/IR mixing allows us to probe high-energy physics by low-energy physics. This observation justifies the use of quantum mechanics to study quantum systems in the presence of minimal lengths. On the other hand, some scenarios have been proposed where the minimal length is related to large extra dimensions [12], to the running coupling constant [13] and to the physics of black hole production [14].

Recently, the Schrödinger equation in the momentum space representation for the harmonic oscillator with minimal lengths in arbitrary dimensions has been solved [1, 2, 15]. The cosmological constant problem and the classical limit of the physics with minimal lengths have also been investigated [16, 17]. On the other hand, the effect of the minimal length on the energy spectrum of the 3D Coulomb potential has been studied in [18, 19] and of the 3D Dirac oscillator using supersymmetric quantum mechanics in [20]. The Casimir force for the electromagnetic field in the presence of the minimal length has also been calculated [21, 22].

In this paper, we solve exactly the Dirac equation in the momentum space representation with an oscillator-like interaction [23, 24], namely the Dirac oscillator in one dimension in the presence of a minimal length. In the case without the minimal length, the Dirac oscillator in one dimension has been investigated by the Green's function technique [25] and by the coherent states approach [26]. On the other hand, the Dirac oscillator in one dimension possesses physical applications in semiconductor physics [27].

The rest of the paper is organized as follows. In section 2, we introduce the main relations of quantum mechanics with generalized Heisenberg uncertainty principle. In section 3, we solve exactly the Dirac equation in one dimension with the oscillator-like interaction in the framework of GUP in the momentum space representation. In section 4, the high-temperature thermal properties are derived. Section 4 is left for concluding remarks.

2. Quantum mechanics with the generalized Heisenberg relation

Following [2], we consider the following simple one-dimensional realization of the position and momentum operators:

$$X = \mathrm{i}\hbar(1+\beta p^2)\frac{\partial}{\partial p}, \qquad P = p, \tag{1}$$

where $\beta \geqslant 0$ is a small parameter. This representation leads to the following generalized commutator and uncertainty relations:

$$[X, P] = \mathrm{i}\hbar(1+\beta p^2), \tag{2}$$

$$\Delta X \Delta P \geqslant \frac{\hbar}{2}[1+\beta(\Delta P)^2]. \tag{3}$$

The peculiarity of (3) is that it exhibits the UV/IR mixing phenomenon which allows us to probe short-distance physics (UV) from the long-distance one (IR). A minimization of (3) with respect to ΔP gives the following minimal length:

$$(\Delta X)_{\min} = \hbar\sqrt{\beta}. \tag{4}$$

This scale, like the UV/IR mixing, reveals the non-local character of the models based on equations (1)–(3). Then we have no localized eigenfunctions in the x-space. So, any eigenvalue problem can be solved by going to the momentum space.

The choice of the constant γ determines only the squeezing degree of the momentum space measure

$$\int \frac{\mathrm{d}p}{(1+\beta p^2)^{1-\frac{\gamma}{\beta}}}|p\rangle\langle p| = \mathbf{1}. \tag{5}$$

Before ending this section, let us point that the energy eigenvalues remain unchanged if we use the following representation of the position and momentum operators: $X = \mathrm{i}\hbar\left[(1+\beta p^2)\frac{\partial}{\partial p} + \gamma f(p)\right]$ and $P = p$, and then we use in the following the simple algebra with $\gamma = 0$.

3. The Dirac oscillator in one dimension

The stationary equation describing the Dirac oscillator in one dimension is given by [23]

$$c\alpha(P - \mathrm{i}\beta m\omega X)\Psi + \beta mc^2\Psi = E\Psi, \tag{6}$$

where m is the rest mass, ω is the classical frequency of the oscillator and $\Psi = \binom{f}{g}$ is a two-component spinor. We point that equation (6) is obtained from the Dirac equation in one dimension by the substitution $P \to P - \mathrm{i}\beta m\omega X$.

Using the following representation of Dirac matrices α and β:

$$\alpha = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

we obtain the following simultaneous equations:

$$c(-\mathrm{i}P + m\omega X)g = (E - mc^2)f, \tag{8}$$

$$c(\mathrm{i}P + m\omega X)f = (E + mc^2)g. \tag{9}$$

In the momentum space realization of the position and momentum operators (1), we have

$$-\mathrm{i}pg + \mathrm{i}\hbar m\omega(1 + \beta p^2)\frac{\partial g}{\partial p} = \frac{(E - mc^2)}{c}f, \tag{10}$$

$$\mathrm{i}pf + \mathrm{i}\hbar m\omega(1 + \beta p^2)\frac{\partial f}{\partial p} = \frac{(E + mc^2)}{c}g. \tag{11}$$

This system gives the following differential equation for the component f:

$$\left[-m^2\omega^2h^2(1 + \beta p^2)^2\frac{\partial^2}{\partial p^2} - 2m^2\omega^2\hbar^2\beta p(1 + \beta p^2)\frac{\partial}{\partial p} + p^2\,(1 - m\omega\hbar\beta)\right] f\,(p)$$
$$= \left(\frac{E^2 - m^2c^4}{c^2} + m\omega\hbar\right) f(p). \tag{12}$$

With the aid of the new variable q defined by $p \in (-\infty, +\infty) \to q \in \left(-\frac{\pi}{2m\omega\hbar\sqrt{\beta}}, +\frac{\pi}{2m\omega\hbar\sqrt{\beta}}\right)$

$$q = \frac{1}{m\omega\hbar\sqrt{\beta}}\arctan p\sqrt{\beta} \tag{13}$$

and setting $\varepsilon = \frac{E^2 - m^2c^4}{c^2} + m\omega\hbar$, we write equation (12) as

$$\left[\frac{\partial^2}{\partial q^2} - \frac{(1 - m\omega\hbar\beta)}{\beta}\tan^2(qm\omega\hbar\sqrt{\beta}) + \varepsilon\right] f(q) = 0. \tag{14}$$

Let us set $f(q) = \upsilon^\lambda h(u)$ where the variables u and υ are given by

$$u = \sin(m\omega\hbar\sqrt{\beta}q), \qquad \upsilon = \cos(m\omega\hbar\sqrt{\beta}q). \tag{15}$$

Then (14) becomes

$$(1 - u^2)h''(u) - (2\lambda + 1)uh'(u)$$
$$+ \left[\left(\lambda(\lambda - 1) - \frac{1 - m\omega\hbar\beta}{(m\omega\hbar\beta)^2}\right)\frac{u^2}{\upsilon^2} - \left(\lambda - \frac{\varepsilon}{m^2\omega^2\hbar^2\beta}\right)\right] h(u) = 0. \tag{16}$$

To reduce (16) to a class of known differential equations, we first eliminate the term proportional to $\frac{u^2}{\upsilon^2}$ by setting

$$\lambda(\lambda - 1) - \frac{1 - m\omega\hbar\beta}{(m\omega\hbar\beta)^2} = 0. \tag{17}$$

This leads to the following expressions for λ:

$$\lambda_1 = \frac{1}{m\omega\hbar\beta}, \qquad \lambda_2 = 1 - \frac{1}{m\omega\hbar\beta}. \tag{18}$$

The solution associated with λ_2 is rejected unless we impose the condition $m\omega\hbar\beta > 1$. However, this condition contradicts the generalized uncertainty principle given by (3). In fact, since $\sqrt{\frac{\hbar}{m\omega}}$ is the characteristic length of the oscillator and $\hbar\sqrt{\beta}$ is the minimal length below it the physics becomes experimentally inaccessible; we must have $m\omega\hbar\beta < 1$. In the following, we set $\lambda_1 = \lambda$.

A polynomial solution to equation (16) is obtained by imposing the following condition:

$$\frac{\varepsilon}{m^2\omega^2\hbar^2\beta} - \lambda = n(n+2\lambda), \tag{19}$$

with n a non-negative integer.

Then equation (16) is written as

$$(1-u^2)h_1''(u) - (2\lambda+1)uh_1'(u) + n(n+2\lambda)h_1(u) = 0, \tag{20}$$

whose solution is given in terms of Gegenbauer's polynomials

$$h\,(u) = NC_n^{\lambda}\,(u)\,, \tag{21}$$

with N a normalization constant. Then the momentum eigenfunctions of the one-dimensional Dirac oscillator in the presence of a minimal length are given by

$$f\,(p) = Nv^{\lambda}C_n^{\lambda}\,(u) \tag{22}$$

and

$$g(p) = \frac{\mathrm{i}c}{\sqrt{\beta}(E+mc^2)}\left(\frac{u}{v} + m\omega\hbar\beta v\frac{\partial}{\partial u}\right)f(u). \tag{23}$$

Using the following property of Gegenbauer's polynomials [28]:

$$\frac{\mathrm{d}}{\mathrm{d}u}C_n^{\lambda}(u) = 2\lambda C_{n-1}^{\lambda+1}(u), \tag{24}$$

we finally obtain

$$f_n(u) = Nv^{\lambda}C_n^{\lambda}(u), \tag{25}$$

$$g_n(u) = \frac{2Nc}{\sqrt{\beta}(E_n+mc^2)}(1-u^2)^{\frac{\lambda+1}{2}}C_{n-1}^{\lambda+1}(u). \tag{26}$$

Returning to the old variable p using the relations

$$u = \frac{p\sqrt{\beta}}{\sqrt{1+\beta p^2}}, \qquad v = \frac{1}{\sqrt{1+\beta p^2}}, \tag{27}$$

we obtain

$$f_n(u) = N(1+\beta p^2)^{-\lambda/2}C_n^{\lambda}\left(\frac{p\sqrt{\beta}}{\sqrt{1+\beta p^2}}\right), \tag{28}$$

$$g_n(u) = \frac{2Nc}{\sqrt{\beta}(E_n+mc^2)}(1+\beta p^2)^{-\lambda-1}C_{n-1}^{\lambda+1}\left(\frac{p\sqrt{\beta}}{\sqrt{1+\beta p^2}}\right). \tag{29}$$

The normalization constant N is calculated from the normalization condition which follows from the modified closure relation (5)

$$\int \frac{\mathrm{d}p}{1+\beta p^2}[|f_n|^2 + |g_n|^2] = 1. \tag{30}$$

Inserting (28) and (29) in (30) and using the following identity [28]:

$$\int_{-1}^{+1} \mathrm{d}u(1-u^2)^{\nu-\frac{1}{2}}\left(C_n^{\nu}(u)\right)^2 = \frac{\pi 2^{1-2\nu}\Gamma(2\nu+n)}{n!(n+\nu)[\Gamma(\nu)]^2}, \tag{31}$$

we obtain

$$N = \frac{2^{\lambda}\beta^{\frac{1}{4}}}{\sqrt{2\pi}}\left[\frac{\Gamma(2\lambda+n)}{n!(n+\lambda)[\Gamma(\lambda)]^2} + \frac{c^2}{\beta(E_n+mc^2)^2}\frac{\Gamma(2\lambda+n+1)}{(n-1)!(n+\lambda)[\Gamma(\lambda+1)]^2}\right]^{-\frac{1}{2}}. \tag{32}$$

Let us now check if the wavefunctions given by (28) and (29) are physically acceptable. Following [2], we must have

$$\langle p^2\rangle = \int \frac{\mathrm{d}p}{1+\beta p^2} p^2[|f_n|^2 + |g_n|^2] < \infty \tag{33}$$

or

$$\int \frac{\mathrm{d}p}{1+\beta p^2} p^2\,|f_n|^2 < \infty \qquad \text{and} \qquad \int \frac{\mathrm{d}p}{1+\beta p^2} p^2\,|g_n|^2 < \infty. \tag{34}$$

In fact, for the small component, the convergence is obvious since λ is positive and g_n behaves like $p^{-2\lambda-2}$. For the large component, we have $f_n \sim p^{-2\lambda}$ and then the convergence criterion requires $\lambda > \frac{1}{2}$. This leads to the following admissible minimal characteristic length of the Dirac oscillator:

$$l_{\text{min}} = \sqrt{\frac{\hbar}{m\omega}} = \frac{(\Delta X)_{\text{min}}}{\sqrt{2}}. \tag{35}$$

This is an expected result since the concept of minimal deals with quantum mechanical extended objects.

The energy spectrum is extracted from (19) which leads to

$$\varepsilon_n = m^2\omega^2\hbar^2\beta(n^2 + (2n+1)\lambda). \tag{36}$$

Using the expression of λ given by equation (18) and ε_n, we obtain

$$E_n = \pm mc^2\sqrt{1+\beta\frac{\omega^2\hbar^2 n^2}{c^2} + 2n\frac{\omega\hbar}{mc^2}}, \qquad n = 0, 1, 2, \ldots. \tag{37}$$

Expanding to first order in β, we obtain

$$E_n = \pm mc^2\sqrt{1+2n\frac{\hbar\omega}{mc^2}}\left[1 \pm \frac{\beta\hbar^2\omega^2}{2c^2}\frac{n^2}{1+2n\frac{\hbar\omega}{mc^2}}\right]. \tag{38}$$

The first term in (38) is the energy spectrum of the usual one-dimensional Dirac oscillator and the second term represents the correction due to the presence of the minimal length. Here we note the dependence on n^2 which is a feature of hard confinement. This is a natural consequence since our original problem is mapped to the motion of a point particle near the surface of a sphere which is in essence a motion in potential wells. In our case, the boundaries of the well are placed at $\pm\frac{\pi}{2m\hbar\omega\sqrt{\beta}}$. An other interesting property of the energy levels given by (37) is that the energy level spacing becomes constant for large n

$$\lim_{n\to\infty} |\Delta E_n| = \hbar\omega mc\sqrt{\beta}. \tag{39}$$

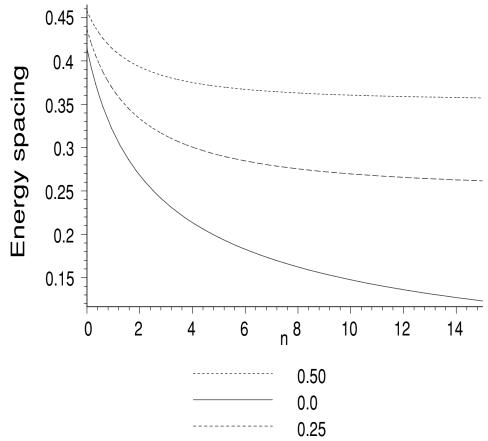

Figure 1. Plot of energy levels spacing ΔE_n versus the quantum number n for different values of the minimal length $\sqrt{\beta}$.

This means that the energy continuum, for large n, of the Dirac oscillator without the minimal length disappears in the presence of the minimal length and that, in this case, the behaviour of the Dirac oscillator can be described for large n by a non-relativistic harmonic oscillator with frequency $\Omega = \omega mc\sqrt{\beta}$. The variation of ΔE_n for different values of the minimal length $\hbar\sqrt{\beta}$ is shown in figure 1.

The non-relativistic limit is obtained, as in the usual case, by setting $E = mc^2 + E_{nr}$ with the assumption that $mc^2 \gg E_{nr}$. A Taylor expansion of (37) gives

$$E_n \approx mc^2 + n\hbar\omega\left(1 + \frac{\beta\hbar\omega mn}{2}\right) - \frac{n^2\hbar^2\omega^2}{2mc^2}\left(1 + \frac{\beta\hbar\omega mn}{2}\right)^2. \tag{40}$$

It is clear that besides the rest energy of the particle, the second and third terms represent, respectively, the energy of the non-relativistic oscillator and the relativistic correction both in the presence of the minimal length.

The corrections to the ordinary harmonic oscillator are obtained by setting $\beta = 0$ in (40) or directly by setting $\beta = \frac{1}{m^2c^2}$ in (37) followed by a Taylor expansion. The last relation implies $(\Delta X)_{\min} = \frac{\hbar c}{mc^2}$.

Let us now study the limit $\beta \to 0$. In this limit, we have $\lambda \to \infty$. Using the following relations [28]:

$$\lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}}\left(x\sqrt{\frac{2}{\lambda}}\right) = \frac{2^{-\frac{n}{2}}}{n!}H_n(x), \qquad \lim_{\lambda\to\infty} \frac{\Gamma(\lambda + a)}{\Gamma(\lambda)}\,\mathrm{e}^{-a\ln\lambda} = 1, \tag{41}$$

the doubling formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) \tag{42}$$

and observing that (to $\mathcal{O}(\beta^2)$)

$$(1 + \beta p^2)^{-\frac{\lambda}{2}} = \exp\left(-\frac{p^2}{2m\hbar\omega}\right), \tag{43}$$

we obtain

$$f_n(p) = \pm \frac{(E_n + mc^2)^2}{c^2} \left[2^{2n} n! \sqrt{\pi m\hbar\omega} \left(\frac{(E_n + mc^2)^2}{c^2} + 2nm\hbar\omega \right) \right]^{-\frac{1}{2}} \times \exp\left(-\frac{p^2}{2m\hbar\omega} \right) H_n \left(\sqrt{\frac{1}{m\hbar\omega}} p \right), \tag{44}$$

$$g_n(p) = \pm \sqrt{n\sqrt{m\hbar\omega}} \left[2^{2n-1} \sqrt{\pi} (n-1)! \left(\frac{(E_n + mc^2)^2}{c^2} + 2nm\hbar\omega \right) \right]^{-\frac{1}{2}} \times \exp\left(-\frac{p^2}{2m\hbar\omega} \right) H_{n-1} \left(\sqrt{\frac{1}{m\hbar\omega}} p \right). \tag{45}$$

These are the momentum space eigenfunctions of the Dirac oscillator without the presence of the minimal length and coincide with those obtained directly from the usual Dirac equation with the oscillator-like interaction.

4. Statistical properties

The partition function of the Dirac oscillator, at a temperature T, in the presence of a minimal length is given by

$$Z_{\tilde{\beta}} = \sum_{n=0}^{\infty} \mathrm{e}^{-E_n/kT} = \sum_{n=0}^{\infty} \exp\left(-\tilde{\beta} mc^2 \sqrt{1 + \frac{\beta\omega^2\hbar^2 n^2}{c^2} + \frac{2n\omega\hbar}{mc^2}} \right). \tag{46}$$

To avoid confusion with notation, we have set $\tilde{\beta} = 1/kT$. The computation of the summation over n is performed with the aid of Euler's formula given by

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\,\mathrm{d}x - \sum_{p=1}^{\infty} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0), \tag{47}$$

where B_{2p} are Bernoulli's numbers and $f^{(2p-1)}(0)$ are derivatives of the function $f(x)$ at $x = 0$.

Setting $\gamma = \tilde{\beta} mc^2$ and $y = \sqrt{1 + \frac{\beta\omega^2\hbar^2 n^2}{c^2} + \frac{2n\omega\hbar}{mc^2}}$, the integral over x in (47) is then given by

$$J = \frac{\left(\frac{mc^2}{\hbar\omega}\right)}{\sqrt{1 - \beta m^2 c^2}} \int_1^{\infty} \mathrm{d}y y \left(1 - \frac{\beta m^2 c^2}{1 - \beta m^2 c^2} y^2 \right)^{-\frac{1}{2}} \mathrm{e}^{-\gamma y}. \tag{48}$$

Using the power series of the square root, the integral can be evaluated with the result

$$J = \frac{\left(\frac{mc^2}{\hbar\omega}\right)}{\sqrt{1 - \beta m^2 c^2}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{\beta m^2 c^2}{1 - \beta m^2 c^2} \right)^n \times \left[\frac{\Gamma(2n+2)}{\gamma^{2n+2}} - \frac{\mathrm{e}^{-\gamma}}{(2n+2)} \Phi(1, 2n+2; \gamma) \right]. \tag{49}$$

In the high-temperature regime $\gamma < 1$, the contributions of the first and third terms in (47) and that of the second term of (49) are negligible compared to the term $\frac{1}{\gamma^{2n+2}}$. Then we obtain

$$Z_{\tilde{\beta}} \simeq \frac{\left(\frac{mc^2}{\hbar\omega}\right)}{(\tilde{\beta} mc^2)^2 \sqrt{1 - \beta m^2 c^2}} \sum_{n=0}^{\infty} (-1)^n \Gamma(2n+2) \frac{(2n-1)!!}{(2n)!!} \alpha^n, \tag{50}$$

with

$$\alpha = \frac{1}{(\tilde{\beta}mc^2)^2}\left(\frac{\beta m^2c^2}{1-\beta m^2c^2}\right). \quad (51)$$

At this stage, we show that for high-temperature expansion α is a small parameter. In fact, we have $1 > \beta m^2c^2$ and α is given by

$$\alpha = \left(\frac{(\Delta X)_{\rm min}}{l_{\rm th}}\right)^2, \quad (52)$$

where $l_{\rm th} = \frac{\hbar c}{kT}$ represents the thermal wavelength obtained at high temperatures. This wavelength is a characteristic length of the system and in order to be experimentally accessible must be greater than the minimal length, by virtue of the generalized uncertainty principle (3). Then α is, as expected, a small parameter.

Then retaining only terms to first order in the deformation parameter β, we obtain

$$Z_{\tilde{\beta}} \simeq \frac{(kT)^2}{\hbar\omega mc^2} - \frac{3\beta(kT)^4}{\hbar\omega mc^4}\left(1-\frac{1}{6}\left(\frac{mc^2}{kT}\right)^2\right). \quad (53)$$

Using the fact that $\left(1-\frac{1}{6}\left(\frac{mc^2}{kT}\right)^2\right) \approx 1$, we finally obtain the high-temperature expansion of the partition function

$$Z_{\tilde{\beta}} \simeq \frac{(kT)^2}{\hbar\omega mc^2} - \frac{3\beta(kT)^4}{\hbar\omega mc^4}. \quad (54)$$

The first term is the partition function for the Dirac oscillator without the minimal length [29], while the second term is the contribution coming from the perturbation of the space by the presence of the minimal length.

From (54), we deduce the following constraint on the minimal distance:

$$(\Delta X)_{\rm min} \leqslant \frac{l_{\rm th}}{\sqrt{3}}. \quad (55)$$

The mean energy defined by $U = kT^2\frac{\partial \ln Z}{\partial T}$ is then given by

$$U \simeq 2kT\left[1-3\left(\frac{(\Delta X)_{\rm min}}{l_{\rm th}}\right)^2\right], \quad (56)$$

while the heat capacity $C = \frac{\partial U}{\partial T}$ is

$$C \simeq 2k\left[1-9\left(\frac{(\Delta X)_{\rm min}}{l_{\rm th}}\right)^2\right]. \quad (57)$$

In the limit $\beta = 0$, i.e. a vanishing minimal length, (56) and (57) reduce to mean energy and heat capacity of the usual Dirac oscillator given, respectively, by $2kT$ and $2k$. Then, at high temperatures, the mean energy and the heat capacity with the minimal length are weaker than those in the case without the minimal length.

From (57) we extract a stronger constraint on the minimal length than that given by (55)

$$(\Delta X)_{\rm min} \leqslant \frac{l_{\rm th}}{3}. \quad (58)$$

Finally, let us note that the non-relativistic harmonic oscillator is used as a model for describing the quarks' confinement in mesons and baryons [30], while the Dirac oscillator is expected to give a good description of the confinement in heavy quark systems [31]. It was also pointed that the thermodynamic properties of the one-dimensional Dirac oscillator are relevant for the description of a quark–gluon plasma [29, 32]. The quark–gluon plasma is expected to be produced by ultrarelativistic collisions of heavy ions. In the latter process, a hot dense hadron gas undergoes an evolution to a thermodynamic equilibrium followed by an essentially one-dimensional adiabatic cooling [34]. This observation justifies the use of the one-dimensional Dirac oscillator in a thermal bath to describe the thermodynamic properties of such a process. The transformation of the hadron gas into a quark–gluon plasma, known as the deconfinement phase, occurs above a temperature T_c. QCD lattice calculation shows that the deconfinement phase occurs at $T_c \simeq 180$ MeV [33–35]. Using T_c in (58), we obtain the following upper bound for the minimal length:

$$(\Delta X)_{\min} \lesssim 0.42\,\text{fm}. \tag{59}$$

Let us observe that this bound is consistent with previous ones [18, 19].

5. Conclusion

In this paper, using the momentum space representation, we have solved exactly the Dirac equation with an oscillator-like interaction in one dimension in the presence of a minimal length. With the aid of appropriate variable transformations, we mapped the problem to that of a point particle in a symmetric potential well with boundaries $\pm\frac{\pi}{2m\omega\sqrt{\beta}}$. Then we obtain the energy eigenvalues and eigenfunctions. Unlike the usual Dirac oscillator in one dimension, the energy levels share a dependence on n^2 like the energy levels of a particle confined in a potential well and the energy level spacing becomes constant for large n, exactly like a usual non-relativistic harmonic oscillator. The validity of the obtained results is checked by rederiving the energy levels and the momentum wavefunctions of the usual one-dimensional Dirac oscillator obtained in the limit $\beta \to 0$. The Dirac oscillator with a minimal length in a thermal bath is also investigated. In the high-temperatures regime, the mean energy and the heat capacity are weaker, due to the presence of the minimal length, than the ordinary ones of the ordinary Dirac oscillator. Finally, using the QCD lattice calculation of the temperature of a quark–gluon plasma, we have obtained an upper bound for the minimal length consistent with previous ones in the literature.

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